Problems on General Relativity: 5
the New Year set

January 2, 2021

Problem 1. Consider a Killing vector X of a metric tensor g, that is such that

$$\mathcal{L}_X g = 0$$

or in other words

$$\nabla_i X_j + \nabla_j X_i = 0.$$

Show, that the acceleration of an observer in rest at an orbit of X is given by the following formula

$$a_j = \frac{1}{2}\frac{e_j\left(X^i X_i\right)}{X^k X_k}.$$

Problem 2. Consider the (Schwarzshild) metric tensor of spherically symmetric, non-rotating body

$$g = -(1 - \frac{r_S}{r})c^2 dt^2 + \frac{dr^2}{1 - \frac{r_S}{r}} + \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

and an observer resting at an orbit of the Killing vector $\frac{\partial}{\partial t}$. Calculate the acceleration 4-vector $a^\mu\partial_\mu$ of the observer, and its length $\sqrt{a_\mu a^\mu}$. Assuming the asymptotic agreement with the Newton's law for large r, show that the radius r_S is related to the mass M of the body as follows

$$r_S = \frac{2GM}{c^2}.$$

Problem 3. Consider the (Kerr) metric tensor of axisymmetric rotating body

$$g = -(1 - \frac{rr_S}{\Sigma})c^2 dt^2 - 2\frac{ar_S r\sin^2\theta}{\Sigma}cdtd\phi + \frac{(r^2+a^2)^2 - (r^2+a^2-rr_S)a^2\sin^2\theta}{\Sigma}\sin^2\Theta d\phi^2 + \frac{\Sigma}{r^2+a^2-rr_S}dr^2 + \Sigma d\theta^2,$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

and an observer resting at an orbit of the Killing vector $\frac{\partial}{\partial t}$. Calculate the acceleration 4-vector $a^\mu\partial_\mu$ of the observer, and its length $\sqrt{a_\mu a^\mu}$. For large r, find the corrections to the Newtons law coming from the angular momentum J of the body, related to the Kerr parameter a as follows

$$a = \frac{J}{Mc}.$$

(Do do not confuse, please, the Kerr parameter a with the acceleration... also a :))

Problem 4. For the metric tensor of Problem 2 find a function f, such that

$$g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} = 0, \qquad \frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial \theta} = 0.$$

Describe a geodesic $\tau \mapsto p(\tau)$ such that

$$\frac{dp(\tau)}{d\tau} = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}.$$

Problem 5. Consider a metric tensor

$$g = g_{ij}e^i e^j$$

the corresponding torsion free and metric connection $\Gamma^i{}_j$, $i, j = 1, ..., n$, and the corresponding Riemann tensor $R^i{}_{jkl}$. Using the symmetries of the Riemann tensor, show that the (Weyl) tensor tensor $C^i{}_{jkl}$ defined by the following equation for $n \geq 3$,

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}(R_{a[c}g_{d]b} - R_{b[c}g_{d]a}) - \frac{2}{(n-1)(n-2)}Rg_{a[c}g_{d]b}$$

satisfies the following identities

$$C_{ijkl} = -C_{jikl} = -C_{ijlk} = C_{klij}, \qquad C^i{}_{jil} = 0.$$

Problem 6. Show that, for $n = 3$ the Weyl tensor defined in Problem 5 is

$$C^i{}_{jkl} = 0.$$

Show, that in that case the following implications are true

$$R_{ij} = 0 \ \Rightarrow \ R_{ijkl} = 0$$

and

$$R_{ij} = 2\Lambda g_{ij} \ \Rightarrow \ R_{abcd} = -2\Lambda(g_{a[c}g_{d]b} + 2g_{b[c}g_{d]a})$$

Problem 6. Show that for $n = 2$ dimensional space (or spacetime), every metric tensor satisfies the vacuum Einstein equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = 0.$$

Clue: use the symmetries of the Riemann tensor.

Problem 7. Using the Bianchi identity

$$\nabla_{[i}R_{jk]mn} = 0$$

satisfied by the Riemann tensor show the following implication

$$R_{ij} = \frac{2}{2-n}\Lambda g_{ij} \ \Rightarrow \ \nabla_i R^i{}_{jkl} = 0.$$